In a nutshell: The 4th-order Runge-Kutta method

Given the initial-value problem (IVP)

$$y^{(1)}(t) = f(t, y(t))$$
$$y(t_0) = y_0$$

we would like to approximate the solution $y(t)$. This algorithm uses Taylor series and iteration. We are given a step size $h > 0$ and a maximum number of steps N and we define $t_k = t_0 + hk$. We want to approximate the solution on the interval $[t_0, t_N]$.

Given an approximation at a point t_k where $y(t_k) \approx y_k$, we will find the approximation y_{k+1} which approximates the solution at $t_k + h = t_{k+1}$.

1. Let $k \leftarrow 0$.
2. If $k = N$, we are finished: we have approximated $y(t_k)$ for $k = 1, \ldots, N$.
3. Let $s_0 = f(t_k, y_k)$
   $s_1 = f(t_k + ½h, y_k + ½hs_0)$,
   $s_2 = f(t_k + ½h, y_k + ½hs_1)$,
   $s_3 = f(t_k + h, \quad y_k + hs_2)$,
   and thus, let $y_{k+1} \leftarrow y_k + h\frac{s_0 + 2s_1 + 2s_2 + s_3}{6}$.
4. Increment k and return to Step 2.

Error analysis

For a single step, the 4th-order Runge-Kutta method is $O(h^5)$ assuming that y_k is exact; however, over multiple steps, where we are using an approximation to estimate the next approximation, the error reduces to $O(h^4)$.